NOTE ON TOTALLY ODD MULTIPLE ZETA VALUES

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Abstract. A partial answer to a conjecture about the rank of the matrix \( C_{N,r} \) introduced by Francis Brown in the study of totally odd multiple zeta values is given.

1. Introduction

In this paper, we give an upper bound of the rank of the matrix \( C_{N,r} \) introduced by Brown [2] in the study of totally odd multiple zeta values. We begin by recalling studies of the matrix \( C_{N,r} \).

The multiple zeta value is defined by

\[
\zeta(n_1, \ldots, n_r) = \sum_{0 < k_1 < \cdots < k_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}
\]

for positive integers \( n_1, \ldots, n_r \in \mathbb{N} \) with \( n_r \geq 2 \). We call \( n_1 + \cdots + n_r \) the weight and \( r \) the depth. In his paper [2, Section 10], Brown introduces the totally odd motivic multiple zeta value of weight \( N \) and depth \( r \)

\[
\zeta_D^m(2n_1 + 1, \ldots, 2n_r + 1) \quad (2n_1 + \cdots + 2n_r = N - r),
\]

which is defined as the motivic multiple zeta value \( \zeta^m(2n_1 + 1, \ldots, 2n_r + 1) \) modulo all motivic multiple zeta values of weight \( N \) and depth \( \leq r - 1 \) and the ideal generated by \( \zeta^m(2) \). For the definition, we refer to [2, §10] and [7, §2]. For positive integers \( N \) and \( r \), let \( \mathbb{I}_{N,r} \) be the set of totally odd indices of weight \( N \) and depth \( r \). The matrix \( C_{N,r} \) is defined in (2.6) whose rows and columns are indexed by \( m = (m_1, \ldots, m_r) \) and \( n = (n_1, \ldots, n_r) \) in \( \mathbb{I}_{N,r} \) and whose coefficients are given by

\[
\partial_{m_2} \cdots \partial_{m_2} \partial_{m_1} \zeta_D^m(n_1, \ldots, n_r) \in \mathbb{Z},
\]

where for odd \( m \in \mathbb{Z}_{\geq 1} \) the \( \partial_m \) is a well-defined derivation corresponding to the canonical generator of the depth-graded motivic Lie algebra \( \mathfrak{d} \) of depth 1 (see [3, Section 2.5] and [2, Section 4] for the definition of \( \mathfrak{d} \)). The homology conjecture of \( \mathfrak{d} \) [2, Conjecture 4] (see also [4]) leads to the following expectations for the matrix \( C_{N,r} \) (see [7, Conjecture 1.1] and [2, Conjecture 5]).

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Conjecture 1.1. i) Rational numbers \( \{a_n\}_{n \in \mathbb{Z}_{N,r}} \) give rise to a linear relation of the form

\[
\sum_{n \in \mathbb{Z}_{N,r}} a_n \zeta_D(n) = 0,
\]

if and only if the column vector \( ^t(a_n)_{n \in \mathbb{Z}_{N,r}} \) is a right annihilator of the matrix \( C_{N,r} \).

Therefore, the rank of the square matrix \( C_{N,r} \) equals the dimension of the \( \mathbb{Q} \)-vector space spanned by all totally odd multiple zeta values of weight \( N \) and depth \( r \).

ii) Let

\[
\mathcal{O}(x) = \frac{x^3}{1-x^2} = x^3 + x^5 + \cdots, \quad \mathcal{S}(x) = \frac{x^{12}}{(1-x^4)(1-x^6)} = x^{12} + x^{16} + \cdots.
\]

Then the generating function of the rank of the matrix \( C_{N,r} \) is given by

\[
1 + \sum_{N,r > 0} \text{rank} C_{N,r} x^N y^r \leq \frac{1}{1 - \mathcal{O}(x)y + \mathcal{S}(x)y^2}.
\]

Conjecture 1.1 i) is true for \( r = 2, 3 \), but still open for \( r \geq 4 \). The first example of linear relations among totally odd multiple zeta value is deduced from

\[
C_{12,2} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
-6 & 0 & 1 & 6 \\
-15 & -14 & 15 & 15 \\
-27 & -42 & 42 & 28
\end{pmatrix}.
\]

The space of right annihilators of \( C_{12,2} \) is generated by the vector \( ^t(14, 75, 84, 0) \), which gives the well-known relation obtained by Gangl, Kaneko and Zagier [5]:

\[
(1.2) \quad 14 \zeta(3,9) + 75 \zeta(5,7) + 84 \zeta(7,5) \equiv 0 \pmod{\mathbb{Q}\zeta(12)}.
\]

As for Conjecture 1.1 ii), it was shown by [1] that the equality (1.1) on the coefficient of \( y^2 \) holds. We also know by Goncharov [6, Theorem 2.5] that \( \text{rank} C_{N,3} \) is bounded by the coefficient of \( x^N y^3 \) in the Taylor expansion of the right hand-side of (1.1) at \( x = y = 0 \). In this paper, we give a partial answer to Conjecture 1.1 ii).

Theorem 1.2. We have

\[
1 + \sum_{N,r > 0} \text{rank} C_{N,r} x^N y^r \leq \frac{1}{1 - \mathcal{O}(x)y + \mathcal{S}(x)y^2},
\]

where \( \sum a_{N,r} x^N y^r \leq \sum b_{N,r} x^N y^r \) means \( a_{N,r} \leq b_{N,r} \) for any \( N, r \).

The principle of our proof is to relate left annihilators of the square matrix \( C_{N,r} \) with the restricted even period polynomials.

The contents of the paper are as follows. In Section 2, the matrix \( C_{N,r} \) is defined. We recall some properties of the matrix \( C_{N,r} \) from [7]. Section 3 is devoted to the proof of Theorem 1.2.
2. Preliminaries

2.1. Notations. We call a tuple of positive integers \( n = (n_1, \ldots, n_r) \) an index. We define the weight \( \mathrm{wt}(n) \) and the depth \( \mathrm{dep}(n) \) of an index \( n = (n_1, \ldots, n_r) \) by \( \mathrm{wt}(n) = n_1 + \cdots + n_r \) and \( \mathrm{dep}(n) = r \), respectively. For an index \( n = (n_1, \ldots, n_r) \), write

\[
x^{n-1} = x_1^{n_1-1} \cdots x_r^{n_r-1}.
\]

Let \( \mathbb{I}_{N,r} \) be the set of totally odd indices of weight \( N \) and depth \( r \):

\[
\mathbb{I}_{N,r} = \{ n = (n_1, \ldots, n_r) \in \mathbb{N}^r \mid \mathrm{wt}(n) = N, n_1, \ldots, n_r \geq 3 : \text{odd} \}.
\]

2.2. Linearized Ihara action. We denote by \( \circ \) the linearised Ihara action (see [2] and [7, §3.1]). It is given by the formula

\[
f \circ g(x_1, \ldots, x_{r+s}) = \sum_{i=0}^s f(x_{i+1} - x_i, \ldots, x_{i+r} - x_i)g(x_1, \ldots, x_i, x_{i+r+1}, \ldots, x_{r+s})
\]

\[
+ (-1)^{\deg(f) + r} \sum_{i=1}^s f(x_{i+r} - x_i, \ldots, x - x_{i+r})g(x_1, \ldots, x_{i-1}, x_{i+r}, \ldots, x_{r+s})
\]

for homogeneous polynomials \( f(x_1, \ldots, x_r) \) and \( g(x_1, \ldots, x_s) \), where \( x_0 = 0 \).

We denote by \( \sigma_r^{(i)} \) (\( 1 \leq i \leq r-1 \)) the following change of variables:

\[
f(x_1, \ldots, x_r) | \sigma_r^{(i)} = f(x_{i+1} - x_i, x_1, \ldots, x_i, x_{i+2}, \ldots, x_r)
\]

\[
- f(x_{i+1} - x_i, x_1, \ldots, x_i, x_{i-1}, x_{i+1}, \ldots, x_r).
\]

We regard \( \sigma_r^{(i)} \) as an element of the group ring \( \mathbb{Z}[	ext{GL}_r(\mathbb{Z})] \). Write \( \sigma_r = \sum_{i=1}^{r-1} \sigma_r^{(i)} \). By (2.1) we have

\[
x_1^{m_1-1} \circ (x_1^{m_2-1} \cdots x_r^{m_r-1}) = x^{m-1} | (1 + \sigma_r)
\]

for any \( m \in \mathbb{N}^r \), where \( f | (1 + \sigma_r^{(i)} + \sigma_r^{(j)}) \) means \( f + f | \sigma_r^{(i)} + f | \sigma_r^{(j)} \).

2.3. Matrices. Following [7, Eq. (3.5) and Definition 2.3], we now define the matrices \( E_{N,r}^{(q)} \) and \( C_{N,r} \) (see [2, Eq. (10.2)] for the original).

For indices \( m = (m_1, \ldots, m_r) \) and \( n = (n_1, \ldots, n_r) \), let us define \( \delta^{(m)}(n) \) as the Kronecker delta given by

\[
\delta^{(m)}(n) = \begin{cases} 1 & \text{if } m_i = n_i \text{ for all } i \in \{1, \ldots, r\} \\ 0 & \text{otherwise} \end{cases}
\]

For indices \( m \) and \( n \) of depth \( r \geq 2 \), we define the integer \( e^{(m)}(n) \) by

\[
x^{m-1} | (1 + \sigma_r) = \sum_{\substack{n \in \mathbb{N}^r \\mathrm{wt}(n) = \mathrm{wt}(m)}} e^{(m)}(n)x^{n-1}.
\]
We set $e^{(m)}_{n_1} = \delta^{(m)}_{n_1}$. For the explicit formula of the integer $e^{(m)}_{n}$ we refer the reader to [7, Lemma 3.1]. We define the integer $c^{(m)}_{n}$ by

$$x_1^{m_{1}-1} \odot (\cdots \odot (x_1^{m_{r}-1} \odot x_1^{m_{r+1}-1}) \cdots ) = \sum_{n \in \mathbb{N}, \text{wt}(n) = \text{wt}(m)} c^{(m)}_{n} x^{n-1}$$

for each index $m = (m_1, \ldots, m_r)$ with $r \geq 2$. We also let $c^{(m)}_{n_1} = \delta^{(m)}_{n_1}$.

**Definition 2.1.** For positive integers $N, r, q$ with $1 \leq q \leq r$, we define the $|\mathbb{I}_{N,r}| \times |\mathbb{I}_{N,r}|$ matrices $E^{(q)}_{N,r}$ and $C_{N,r}$ by

$$E^{(q)}_{N,r} = \left( \delta^{(m_1, \ldots, m_{q}-1)}_{n_1, \ldots, n_{q}} \cdot e^{(m_{q+1}, \ldots, m_{r})}_{n_{q+1}, \ldots, n_{r}} \right)_{(m_1, \ldots, m_r) \in \mathbb{I}_{N,r}, (n_1, \ldots, n_r) \in \mathbb{I}_{N,r}}$$

and

$$C_{N,r} = \left( c^{(m)}_{n} \right)_{m \in \mathbb{I}_{N,r}, n \in \mathbb{I}_{N,r}}$$

where rows and columns are indexed by $m$ and $n$ in the set $\mathbb{I}_{N,r}$. It is understood that the matrices $E^{(q)}_{N,r}$ and $C_{N,r}$ are an empty matrix when $|\mathbb{I}_{N,r}| = 0$ (i.e. rank $C_{N,r} = 0$ and $\ker C_{N,r} = \{0\}$).

The matrix $E^{(1)}_{N,r}$ is the identity matrix when $|\mathbb{I}_{N,r}| \neq 0$. One can write the matrix $C_{N,r}$ in the form

$$C_{N,r} = E^{(1)}_{N,r} E^{(2)}_{N,r} \cdots E^{(r-1)}_{N,r} \cdot E^{(r)}_{N,r}$$

for positive integers $N, r$ (see [7, Proposition 3.3]).

2.4. **Linear maps.** For positive integers $N, r \geq 1$, let $V_{N,r}$ denote the $|\mathbb{I}_{N,r}|$-dimensional vector space over $\mathbb{Q}$ of row vectors $(a_n)_{n \in \mathbb{I}_{N,r}}$ indexed by totally odd indices $n \in \mathbb{I}_{N,r}$ with rational coefficients:

$$V_{N,r} = \{(a_n)_{n \in \mathbb{I}_{N,r}} \mid a_n \in \mathbb{Q}\}.$$ 

If $|\mathbb{I}_{N,r}| = 0$, we set $V_{N,r} = \{0\}$. The matrix $C_{N,r}$ is viewed as the linear maps on $V_{N,r}$ in the following manner (see also [7, §2.2]):

$$C_{N,r} : V_{N,r} \longrightarrow V_{N,r} \quad v = (a_n)_{n \in \mathbb{I}_{N,r}} \longmapsto v \cdot C_{N,r} = \left( \sum_{m \in \mathbb{I}_{N,r}} a_{m} c^{(m)}_{n} \right)_{n \in \mathbb{I}_{N,r}}.$$ 

For any subspace $W$ of $V_{N,r}$, we denote the image of $W$ under the map $C_{N,r}$ by

$$WC_{N,r} = \{v \cdot C_{N,r} \mid v \in W\} \subset V_{N,r}.$$ 

The $\mathbb{Q}$-vector subspace of $V_{N,r}$ of left annihilators of the matrix $C_{N,r}$ is denoted by

$$\ker C_{N,r} = \{v \in V_{N,r} \mid v \cdot C_{N,r} = 0\}.$$
We also apply the above convention to the matrices $E_{N,r}^{(q)}$.

2.5. **Tensor product.** For convenience we view $V_{N,r} \otimes Q V_{M,s}$ as a subspace of $V_{N+M,r+s}$ in the following manner. For two row vectors $(a_n)_{n \in I_{N,r}} \in V_{N,r}$ and $(b_n)_{n \in I_{M,s}} \in V_{M,s}$, the coefficient $c_{n_1,\ldots,n_{r+s}}$ of the row vector

$$(c_n)_{n \in I_{N+M,r+s}} = (a_n)_{n \in I_{N,r}} \otimes (b_n)_{n \in I_{M,s}} \in V_{N+M,r+s}$$

is defined by

$$c_{n_1,\ldots,n_{r+s}} = \begin{cases} a_{n_1,\ldots,n_r} b_{n_{r+1},\ldots,n_{r+s}} & \text{if } (n_1,\ldots,n_r) \in I_{N,r} \text{ and } (n_{r+1},\ldots,n_{r+s}) \in I_{M,s} \\ 0 & \text{otherwise} \end{cases}$$

for each $(n_1,\ldots,n_{r+s}) \in I_{N+M,r+s}$. Note that for $n \in I_{N+M,r+s}$ the above $c_n$ can be obtained from the coefficient of $x^{n-1}$ in $f(x_1,\ldots,x_r)g(x_{r+1},\ldots,x_{r+s})$, where we write $f(x_1,\ldots,x_r) = \sum_{n \in I_{N,r}} a_n x^{n-1}$ and $g(x_1,\ldots,x_r) = \sum_{n \in I_{M,s}} b_n x^{n-1}$. With this notation, we let

$$V_{N,r} \otimes V_{M,s} = \{ v \otimes w \mid v \in V_{N,r}, w \in V_{M,s} \},$$

which is a $Q$-subvector space of $V_{N+M,r+s}$. We remark that since $V_{N,r} = \bigoplus_{N_1+N_2=N} (V_{N_1,r-q} \otimes V_{N_2,q})$ holds for any $0 < q < r$, by definition (2.5) we have

$$(2.8) \quad V_{N,r} E_{N,r}^{(q)} = \bigoplus_{N_1+N_2=N} (V_{N_1,r-q} \otimes V_{N_2,q} E_{N_2,q}^{(q)}).$$

2.6. **Restricted even period polynomials.** Let

$$W_{N,2} = \ker E_{N,2}^{(2)} = \{ v \in V_{N,2} \mid v \cdot E_{N,2}^{(2)} = 0 \}.$$

It was shown by [1] (see also [7, Proposition 3.4]) that the row vector $(a_n)_{n \in I_{N,2}} \in V_{N,2}$ is an element in $W_{N,2}$ if and only if the polynomial $f(x_1, x_2) = \sum_{m \in I_{N,2}} a_m x^{m-1}$ satisfies

$$0 = f(x_1, x_2) - f(x_2 - x_1, x_2) + f(x_2 - x_1, x_1) = f(x_1, x_2) \Big/ (1 + \sigma_2^{(1)}).$$

Thus, the space $W_{N,2}$ is isomorphic to the $Q$-vector space of restricted even period polynomials of degree $N - 2$ (see [5, §5]). We therefore find that

$$(2.9) \quad \sum_{N>0} \dim W_{N,2} x^N = \mathbb{S}(x).$$

For positive integers $N, r, p$ with $2 \leq p \leq r - 2$, we consider the subspace

$$W^{(p)}_{N,r} = \bigoplus_{N_1+N_2+N_3=N} (V_{N_1,p-1} \otimes W_{N_2,2} \otimes V_{N_3,r-p-1})$$

of $V_{N,r} = \bigoplus_{N_1+N_2+N_3=N} \left( V_{N_1,p-1} \otimes V_{N_2,2} \otimes V_{N_3,r-p-1} \right)$. We also consider

$$W^{(1)}_{N,r} = \bigoplus_{N_1+N_2=N} (W_{N_1,2} \otimes V_{N_2,r-2}) \quad \text{and} \quad W^{(r-1)}_{N,r} = \bigoplus_{N_1+N_2=N} (V_{N_1,r-2} \otimes W_{N_2,2}).$$
For any positive integers $N, q$ satisfying $1 \leq q \leq r - p - 1$, it follows that

$$W_{N,r}^{(p)} E_{N,r}^{(q)} = \bigoplus_{N_1+N_2+N_3=N} \left( V_{N_1,p-1} \otimes W_{N_2,2} \otimes V_{N_3,r-p-1}E_{N_3,r-p-1}^{(q)} \right).$$

Hence, by (2.8) we have

$$W_{N,r}^{(p)} E_{N,r}^{(q)} \subset W_{N,r}^{(p)} \quad (1 \leq q \leq r - p - 1).$$

3. Proof of Theorem 1.2

Since rank $C_{N,r} = |\mathbb{I}_{N,r}| - \dim \ker C_{N,r}$, we give a lower bound of $\dim \ker C_{N,r}$ in order to prove Theorem 1.2. Note that since $\sum_{N>0} |\mathbb{I}_{N,r}|x^N = \mathcal{O}(x)^r$, it suffices to show the inequality

$$\sum_{N,r \geq 2} \dim \ker C_{N,r} x^N y^r \geq \frac{S(x)y^2}{(1 - \mathcal{O}(x)y)(1 - \mathcal{O}(x)y + S(x)y^2)}.$$

We begin with the following proposition.

**Proposition 3.1.** For any positive integers $N, r \geq 2$, we have

$$\sum_{p=1}^{r-1} W_{N,r}^{(p)} \subset \ker C_{N,r}.$$

**Proof.** For $1 \leq p \leq r - 1$, by (2.7) and (2.10) we have

$$W_{N,r}^{(p)} C_{N,r} \subset W_{N,r}^{(p)} E_{N,r}^{(r-p)} E_{N,r}^{(r-p+1)} \cdots E_{N,r}^{(r)}.$$

We now prove the inclusion

$$W_{N,r}^{(p)} E_{N,r}^{(r-p)} \subset \ker E_{N,r}^{(r-p+1)},$$

from which Proposition 3.1 follows. Since $W_{N,r}^{(1)} = \bigoplus_{N_1+N_2=N} (W_{N_1,2} \otimes V_{N_2,r-2})$, for $2 \leq p \leq r - 1$ one can write $W_{N,r}^{(p)}$ in the form

$$W_{N,r}^{(p)} = \bigoplus_{N_1+N_2=N} \left( V_{N_1,p-1} \otimes W_{N_2,r-p+1}^{(1)} \right).$$

Applying the matrix $E_{N,r}^{(r-p)}$ to the space (3.3) we have

$$W_{N,r}^{(p)} E_{N,r}^{(r-p)} = \bigoplus_{N_1+N_2=N} \left( V_{N_1,p-1} \otimes W_{N_2,r-p+1}^{(1)} E_{N_2,r-p+1}^{(r-p)} \right).$$

Hence, what is left is to show that the inclusion $W_{N,q}^{(1)} E_{N,q}^{(q-1)} \subset \ker E_{N,q}^{(q)}$ holds for all $N, q \geq 2$. The case $q = 2$ is immediate from the definition. We assume $q = r \geq 3$.

Let $(a_n)_{n \in \mathbb{N}_r}$ be an element in $W_{N,r}^{(1)}$. We write $(b_n)_{n \in \mathbb{N}_r} = (a_n)_{n \in \mathbb{N}_r} E_{N,r}^{(r-1)} E_{N,r}^{(r)}$ and set $p(x_1, \ldots, x_r) = \sum_{m \in \mathbb{N}_r} a_m x_1^{m-1}$. Using (2.2), one can easily check that for any
For any integers \( n \in \mathbb{N}_r \), we have

\[
b_n = \text{coefficient of } x^{n-1} \text{ in } \left( \sum_{(m_1, \ldots, m_r) \in \mathbb{N}_r} a_{m_1, \ldots, m_r} x_1^{m_1-1} h_{m_2, \ldots, m_r}(x_2, \ldots, x_r) \right) (1 + \sigma_r),
\]

where we write \( h_{m_1, \ldots, m_r}(x_1, \ldots, x_r) = x_1^{m_1-1} \circ (x_1^{m_2-1} \cdots x_r^{m_r-1}) \). Since

\[
p(x_1, \ldots, x_r) \big| (1 + \sigma_r^{(1)}) = 0,
\]

we have

\[
p(x_1, \ldots, x_r) = p(x_2, x_1, x_3, \ldots, x_r).
\]

With this and (2.2) one can compute

\[
\sum_{(m_1, \ldots, m_r) \in \mathbb{N}_r} a_{m_1, \ldots, m_r} x_1^{m_1-1} h_{m_2, \ldots, m_r}(x_2, \ldots, x_r)
= p(x_1, \ldots, x_r) \big| (1 - \sigma_r^{(2)} - \sigma_r^{(3)} - \cdots - \sigma_r^{(r-1)}) = -p(x_1, \ldots, x_r) | \sigma_r,
\]

where for the first (resp. the last) equality we have used (3.5) (resp. (3.4)). Then, the statement that \( b_n = 0 \) for all \( n \in \mathbb{N}_r \) follows from the equation (see [7, Eq. (3.31)])

\[
p(x_1, \ldots, x_r) | \sigma_r \big| (1 + \sigma_r) = 0,
\]

which completes the proof of (3.2). \(\square\)

In what follows, we compute the dimension of the space \( \sum_{1 \leq p \leq r-1} W_{N,r}^{(p)} \) to give a lower bound of \( \dim \ker C_{N,r} \). By linear algebra, we see that

\[
\dim \left( \sum_{p=1}^{r-1} W_{N,r}^{(p)} \right) = \sum_{l=0}^{r-1} (-1)^{l+1} \sum_{0 < t_1 < \cdots < t_l < r} \dim \left( \bigcap_{j=1}^l W_{N,r}^{(t_j)} \right).
\]

**Lemma 3.2.** For any integers \( 0 < t_1 < \cdots < t_l < r \) we have

\[
\sum_{N > 0} \dim \left( \bigcap_{j=1}^l W_{N,r}^{(t_j)} \right) x^N = \begin{cases} S(x)^l \big| (x)^{-2l} & \text{if } t_j - t_{j-1} \geq 2 \ (2 \leq j \leq l), \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** By [4, Proposition 6.4], we have \( W_{N,3}^{(1)} \cap W_{N,3}^{(2)} = \{0\} \), and hence, the intersection \( \bigcap_{j=1}^l W_{N,r}^{(t_j)} \) is trivial if there exists \( j \in \{2, \ldots, l\} \) such that \( t_j - t_{j-1} = 1 \). If \( t_j - t_{j-1} \geq 2 \) for all \( 2 \leq j \leq l \), since the intersection \( \bigcap_{j=1}^l W_{N,r}^{(t_j)} \) is a subspace of

\[
V_{N,r} = \bigoplus_{N_1 + \cdots + N_l = N} \left( V_{N_1,t_1-1} \otimes \left( \bigotimes_{j=2}^l W_{N_j,t_j-t_{j-1}} \right) \otimes V_{N_{l+1},r-t_{l+1}} \right),
\]

using (3.3) we have

\[
\bigcap_{j=1}^l W_{N,r}^{(t_j)} = \bigoplus_{N_1 + \cdots + N_l = N} \left( V_{N_1,t_1-1} \otimes \left( \bigotimes_{j=2}^l W_{N_j,t_j-t_{j-1}}^{(1)} \right) \otimes W_{N_{l+1},r-t_{l+1}}^{(1)} \right).
\]
Then the formula is immediate from (2.9).

We are now in a position to prove (3.1).

Proof of Theorem 1.2. It suffices to show

\[
\sum_{N, r \geq 2} \dim \left( \bigcup_{p=1}^{r-1} W_{N, r}^{(p)} \right) x^N y^r = \frac{S(x)y^2}{(1 - \mathcal{O}(x)y)(1 - \mathcal{O}(x)y + S(x)y^2)},
\]

from which by Proposition 3.1 the inequality (3.1) follows. By (3.6) and Lemma 3.2, the left-hand side can be computed as follows:

\[
\sum_{N>0} \sum_{r \geq 2} \dim \left( \bigcup_{p=1}^{r-1} W_{N, r}^{(p)} \right) x^N y^r = \sum_{N>0} \sum_{r \geq 2} \left( \sum_{l=1}^{r-1} (-1)^{l+1} \sum_{0 < t_1 < \ldots < t_l < r} \dim \left( \bigcap_{j=1}^{l} W_{N, r}^{(t_j)} \right) \right) x^N y^r
\]

\[
= \sum_{r \geq 2} \left( \sum_{l=1}^{r/2} (-1)^{l+1} S(x)^l \mathcal{O}(x)^{r-2l} \sum_{0 < t_1 < \ldots < t_l < r} \dim \left( \bigcap_{j=1}^{l} W_{N, r}^{(t_j)} \right) \right) x^N y^r,
\]

where for the last equality we note that \( \dim \left( \bigcap_{j=1}^{l} W_{N, r}^{(t_j)} \right) = 0 \) if \( l > r/2 \) (recall [4, Proposition 6.4]). Letting \( X = \mathcal{O}(x)y, Y = S(x)y^2 \), we have

\[
= \sum_{r \geq 2} \left( \sum_{l=1}^{r/2} (-1)^{l+1} Y^l X^{r-2l} \sum_{0 < t_1 < \ldots < t_l < r} \dim \left( \bigcap_{j=1}^{l} W_{N, r}^{(t_j)} \right) \right) \]

\[
= - \sum_{l \geq 1} (-Y)^l \sum_{r \geq l} X^{2r-2l} \sum_{0 < t_1 < \ldots < t_l < r} \dim \left( \bigcap_{j=1}^{l} W_{N, r}^{(t_j)} \right) \]

\[
- \sum_{l \geq 1} (-Y)^l \sum_{r \geq l} X^{2r+1-2l} \sum_{0 < t_1 < \ldots < t_l < 2r+1} \dim \left( \bigcap_{j=1}^{l} W_{N, r}^{(t_j)} \right) \]

\[
= - \sum_{l \geq 1} (-Y)^l \sum_{r \geq 0} X^r \sum_{0 < t_1 < \ldots < t_l < r+2l} \dim \left( \bigcap_{j=1}^{l} W_{N, r}^{(t_j)} \right) \]

Using \( 1/(1 - X) = \sum_{r \geq 0} X^r \), one can check the identity

\[
\sum_{r \geq 0} X^r \sum_{0 < t_1 < \ldots < t_l < r+2l} \dim \left( \bigcap_{j=1}^{l} W_{N, r}^{(t_j)} \right) \]

\[
= \frac{1}{(1 - X)^{l+1}}.
\]
by induction on \( l \). Hence,
\[
= -\sum_{l \geq 1} \frac{(-Y)^l}{(1 - X)^{l+1}} = -\frac{1}{1 - X} \sum_{l \geq 1} \left( \frac{-Y}{1 - X} \right)^l \\
= -\frac{1}{1 - X} \frac{-Y}{1 - X} = \frac{Y}{(1 - X)(1 - X + Y)},
\]
which completes the proof. \( \square \)

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